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Corrections to the Critical Temperature in 2D Ising Systems with Kac Potentials

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We consider a $d=2$ Ising system with a Kac potential whose mean-field critical temperature is 1. Calling $\gamma>0$ the Kac parameter, we prove that there exists $c^*>0$ so that the true inverse critical temperature $\beta_{cr}(\gamma)>1+b\gamma^2\log\gamma^{-1}$, for any $b<c^*$ and γ correspondingly small. We also show that if $\gamma\to 0$ and $b\to c^*$, suitably, then the correlation functions (normalized and rescaled) converge to those of a non-Gaussian Euclidean field theory.

KEY WORDS: Kac potential; Ising model; critical fluctuations; Euclidean field theory.

In this short communication we report some results we have obtained studying the $d=2$ Ising model with Kac potentials; details of the proofs will follow in an extended version of this note. Recall that the Kac Hamiltonian[11] is

$$H_\gamma(\sigma)=-\tfrac{1}{2}\sum_{x\neq y} J_\gamma(x,y)\,\sigma(x)\,\sigma(y) \tag{1.1}$$

where the spins are $\sigma(x)=\pm 1$, $x\in\mathbb{Z}^d$, and the coupling strength is

$$\begin{aligned} J_\gamma(x,y)&=c_\gamma\gamma^d J(\gamma\,|x-y|), \qquad \sum_{y\neq x} J_\gamma(x,y)=1 \\ \int dr\, J(|r|)&=1, \qquad D:=\int_{\mathbb{R}^2} dr\, J(|r|)\, r^2 \end{aligned} \tag{1.2}$$

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We suppose $J(|r|) \geqslant 0$ and, to fix ideas, we take J "smooth," $J(|r|) > 0$ for $|r| < 1$ and $J(|r|) = 0$ for $|r| \geqslant 1$. Here $\gamma > 0$ is the scaling parameter of the Kac potential and c_γ is a normalization constant that goes to 1 as $\gamma \to 0$.

The topics we are going to discuss are:

1. The corrections to the mean-field (Lebowitz–Penrose) inverse critical temperature $\beta_{\text{cr}}^{\text{LP}}$, namely the deviations from $\beta_{\text{cr}}^{\text{LP}}$ of the true inverse critical temperature $\beta_{\text{cr}}(\gamma)$ for the system with interaction J_γ.
2. The derivation of the ϕ^4 Euclidean field theory when $\gamma \to 0$, with β a suitable function of γ.

We will see in Theorem 1 that $\beta_{\text{cr}}(\gamma)$ is strictly larger than 1 and that the deviations have order $\gamma^2 \log \gamma^{-1}$. In Theorem 2 we prove that the correlation functions of the Ising model suitably scaled and normalized converge as $\gamma \to 0$ to the moments of a non-Gaussian Euclidean field theory.

The corrections to the critical temperature have been studied in ref. 5 for a $d = 3$ Ising system with a specific choice of the Kac potential and the deviation from mean field is $c_0\gamma^2 + o(\gamma^2)$, with c_0 explicitly known. However, to avoid spurious fluctuations of $\beta_{\text{cr}}(\gamma)$ as γ varies, one should compare systems at different γ with the same total interaction strength $\sum J_\gamma(0, x)$. For this reason we have introduced the parameter c_γ in (1.2). This convention is not used in ref. 5, where the total interaction strength is equal to $1 + c_0\gamma^2$. With our normalization (1.2) the result in ref. 5 becomes

$$1 \leqslant \beta_{\text{cr}}(\gamma) \leqslant 1 + o(\gamma^2) \tag{1.3}$$

and there is no correction to the inverse critical temperature of the order of γ^2.

Condition (1.3) shows that $\beta_{\text{cr}}(\gamma) \to 1$ as $\gamma \to 0$. Even if this is generally believed to hold in $d \geqslant 2$, to our knowledge this has not yet been proved. A possible way is to find lower and upper bounds that squeeze $\beta_{\text{cr}}(\gamma)$ to 1 as $\gamma \to 0$. For the upper bound we only have a conjecture:

Conjecture. For any $\beta > 1$ there is $\gamma(\beta) > 0$ so that for all $\gamma \leqslant \gamma(\beta)$, $\beta_{\text{cr}}(\gamma) < \beta$.

A proof of this conjecture exists when reflection positivity may be applied, as in the original Kac potential and in the model considered in ref. 5; see (1.3). In general the proof should follow from a Peierls argument complemented by some large-deviation estimates and bounds on the surface tension. Some progress on the last two issues has been recently obtained.[13,1]

The other side of the bound comes from mean field, as observed in ref. 5. It is known that $\beta_{cr}(\gamma) \geq \beta_{cr}^{LP} = 1$, as we will discuss later. Thus, assuming the validity of the Conjecture, we conclude that $\beta_{cr}(\gamma) \to 1$ as $\gamma \to 0$. The next question concerns the rate of convergence. Our results will show that $\beta_{cr}(\gamma) > 1$ (strictly, for $\gamma > 0$) and that the deviations from 1 are responsible for the Wick regularization and the convergence of the block spin variables to a non-Gaussian Euclidean field theory. Let us be more precise, starting from the first statement:

Theorem 1. For any $b < c^* := 1/(\pi D)$, D as in (1.2), there is $\gamma(b) > 0$ so that for all $0 < \gamma \leq \gamma(b)$

$$\beta_{cr}(\gamma) \geq \beta_b(\gamma) := 1 + b\gamma^2 \log \gamma^{-1} \tag{1.4}$$

This theorem suggests we study the critical behavior by letting $b \to c^*$ as $\gamma \to 0$. The analysis of this limit is in the same spirit of the continuum limit considered, for instance, in the works of Aizenman,[2,3] Sokal,[15] Fröhlich,[9] and Brydges *et al.*[6] and references therein, where they prove convergence to ϕ^4 in $d = 2, 3$ and to a Gaussian field in higher dimensions, using, and exploiting, the relation with Ising systems and statistical mechanics. Our systems do not seem to be included, at least explicitly, in the class considered in the above papers and we have worked out a specific proof that has the advantage of making explicit the relation between the shift of the critical temperature, the origin of the Wick regularization term, and the fluctuation strength.

We study the convergence of the block spin variables by considering the normalized and scaled correlation functions. We denote by $\langle \cdot \rangle_{b,\gamma}$ the expectation with respect to the Gibbs measure with interaction J_γ and inverse temperature $\beta_b(\gamma)$; see (1.4). By Theorem 1 for $\gamma < \gamma(b)$ there is only one Gibbs state and no ambiguity may arise. Sometimes we will write $\langle \cdot \rangle_{\beta,\gamma}$ for the expectation with respect to the Gibbs measure at the inverse temperature β.

From the proof of Theorem 1 it follows that the correlations decay on the scale

$$l_{b,\gamma} = \frac{\gamma^{-2}}{[(c^* - b) \log \gamma^{-1}]^{1/2}} \tag{1.5}$$

Thus, in interaction-length units, the correlation length is

$$\gamma l_{b,\gamma} = \frac{1}{[\beta_{c^*}(\gamma) - \beta_b(\gamma)]^{1/2}} \tag{1.6}$$

This is the result predicted by mean-field theory, because the scaling exponent of the correlation length, in interaction-length units, with respect to the inverse temperature difference is 1/2. Thus the mean-field critical exponent is "correct" when the temperature "is not too close" to critical, i.e., $b < c^*$, as in (1.4). (This remark complements that of Aizenman about the validity of mean field for computing critical exponents in $d > 4$; see the beginning of Section 4 in ref. 3). Notice, however, that Theorem 1 does not tell us that $\beta_{c^*}(\gamma)$ is the true inverse critical temperature; the r.h.s. of (1.6) is only an upper bound to the critical exponent.

On the other hand, we expect a change of behavior when the distance of the inverse temperature from $\beta_{c^*}(\gamma)$ is of the order of γ^2. We in fact prove in Theorem 2 below that our system behaves as a massive local field theory when $b \to c^*$ and $\gamma \to 0$ (suitably) and the value of the mass is proportional to the coefficient multiplying the term in γ^2 of the inverse temperature difference. Therefore, when the inverse temperature varies on the scale γ^2, we expect to recover the n.n. Ising critical exponent.

We introduce the normalized and rescaled correlations

$$S_{b,\gamma}(r_1,..., r_{2n}) = \gamma^{-2n} \langle \sigma(l_{b,\gamma} r_1) \cdots \sigma(l_{b,\gamma} r_{2n}) \rangle_{b,\gamma} \tag{1.7}$$

where $l_{b,\gamma} r_i$ should be replaced by its integer part, and all sites are supposed distinct. We have the following result:

Theorem 2. There is $\varepsilon_0 > 0$ so that the following holds. For any $\varepsilon \leqslant \varepsilon_0$ there is b_γ such that

$$\lim_{\gamma \to 0} [(c^* - b_\gamma) \log \gamma^{-1}]^{1/2} = \varepsilon^{-1} \tag{1.8}$$

and for any n and any distinct $r_1,..., r_{2n}$,

$$\lim_{\gamma \to 0} S_{b,\gamma}(r_1,..., r_{2n}) = S^{(\varepsilon)}(r_1,..., r_{2n}) \tag{1.9}$$

The Schwinger functions $S^{(\varepsilon)}$ are continuous in $\{r_i \neq r_j\}$, and for any test function $\psi(r_1,..., r_{2n})$,

$$\lim \int dr_1 \cdots dr_{2n} \, S_{b,\gamma}(r_1,..., r_{2n}) \, \psi(r_1,..., r_{2n})$$

$$= \int dr_1 \cdots dr_{2n} \, S^{(\varepsilon)}(r_1,..., r_{2n}) \, \psi(r_1,..., r_{2n}) \tag{1.10}$$

The functions $S^{(\varepsilon)}$ satisfy a recursive relation that is also satisfied by the Schwinger functions of the ϕ^4 Euclidean theory with interaction strength $\lambda = 1$ and mass ε^{-1}. In particular, the truncated correlation functions constructed from $S^{(\varepsilon)}$ are not identically 0.

Without entering into the proofs of the two theorems, we just want to outline some points that may be of interest. We recall that by using the Dobrushin's techniques for the uniqueness of Gibbs measures,[7] we get for the Vasserstein distance $R_x(\sigma, \sigma')$ between the conditional probabilities at x given two different boundary conditions σ and σ',

$$R_x(\sigma, \sigma') \leqslant \beta \sum_{y \neq x} J_\gamma(x, y) \, |\sigma(y) - \sigma(y')| \tag{1.11}$$

Recalling (1.2), the Dobrushin uniqueness condition $\beta \sum J_\gamma(x, y) < 1$ is satisfied for all $\beta < 1$ and all γ, hence $\beta_{cr}(\gamma) \geqslant 1$. The same analysis in ref. 7 allows one to derive bounds for the correlation functions, which in the case of the two-point correlation yields

$$\langle \sigma(0) \, \sigma(x) \rangle_{\beta,\gamma} \leqslant \beta \sum_{y \neq 0} J_\gamma(x, y) \langle \sigma(0) \, \sigma(y) \rangle_{\beta,\gamma} + 2\beta J_\gamma(x, 0) \tag{1.12}$$

hence

$$\langle \sigma(0) \, \sigma(x) \rangle_{\beta,\gamma} \leqslant 2[(1 - \beta J_\gamma)^{-1}]_{x,0} \tag{1.13}$$

On the other hand, by the DLR equations,

$$\langle \sigma(0) \, \sigma(x) \rangle_{\beta,\gamma} = \langle \sigma(0) \tanh \beta h_\gamma(x) \rangle_{\beta,\gamma}, \quad h_\gamma(x) = \sum_{y \neq x} J_\gamma(x, y) \, \sigma(y) \tag{1.14}$$

By Taylor expanding the hyperbolic tangent and retaining only the first two terms [as can be rigorously justified using Newmann's Gaussian inequalities,[14] and (1.13)] we get

$$\langle \sigma(0) \, \sigma(x) \rangle_{\beta,\gamma} \approx \beta \langle \sigma(0) \, h_\gamma(x) \rangle_{\beta,\gamma} - \tfrac{1}{3} \beta^3 \langle \sigma(0) \, h_\gamma(x)^3 \rangle_{\beta,\gamma} \tag{1.15}$$

Given any $\beta < 1$, it is not difficult to see, using the previous arguments, that also the last term is of higher order, in the limit as $\gamma \to 0$. We can then solve (1.14) and more generally we find that for any $n \geqslant 1$ and any distinct $r_1, ..., r_{2n}$ (and in any dimension d)

$$\lim_{\gamma \to 0} \gamma^{-2n} \langle \sigma(\gamma^{-1} r_1) \cdots \sigma(\gamma^{-1} r_{2n}) \rangle_{\beta,\gamma} = \sum_{\text{pairings}} \prod_{l=1}^{n} G(r_{i_l} - r_{j_l}) \tag{1.16}$$

where $G = G_1$ and

$$G_t(r) = \frac{1}{2\pi D t} e^{-r^2/2Dt} \tag{1.17}$$

D is as in (1.2). The bound (1.12) is quite accurate when $\beta < 1$ is kept fixed as $\gamma \to 0$, but it starts deviating from the correct one as soon as $\beta = 1 - \gamma^2$. We can rewrite (1.15) as

$$\langle \sigma(0)\, \sigma(x) \rangle_{\beta,\gamma} \approx (\beta - \beta^3 C_{\beta,\gamma}) \langle \sigma(0)\, h_\gamma(x) \rangle_{\beta,\gamma} - \tfrac{1}{3}\beta^3 \langle \sigma(0)\, h_\gamma(x)^3 \rangle^{\mathrm{T}}_{\beta,\gamma} \quad (1.18)$$

where $\langle \cdot \rangle^{\mathrm{T}}_{\beta,\gamma}$ is the truncated correlation function and

$$C_{\beta,\gamma} = \langle h_\gamma(x)^2 \rangle_{\beta,\gamma} = \gamma^2 \log \gamma^{-1} \left[\frac{1}{\pi D} + O\left(\frac{1}{\log \gamma^{-1}} \right) \right],$$

$$\text{for} \quad \beta = \beta_b(\gamma), \quad b < c^* \quad (1.19)$$

[An upper bound of the right order at $\beta = 1 - \gamma^2$ follows directly from (1.12).] Because of the Gaussian structure of the correlations, evidenced by (1.16), we should have, and we actually do have, that the truncated correlation function may be neglected as $\gamma \to 0$.

From (1.18) and (1.19) and after some computations, we then see that the effective temperature is

$$\beta - \beta^3 C_{\beta,\gamma} = \beta - c^* \gamma^2 \log \gamma^{-1} + O(\gamma^2) \quad (1.20)$$

This establishes the relation between the temperature shift and the Wick regularization (i.e., replacing a correlation function by the truncated one). The same mechanism was observed in ref. 5 in a $d = 3$ lattice approximation of ϕ^4; see also ref. 6.

The hard part in the proof of Theorems 1 and 2 is to extend the validity of the above considerations beyond $\beta = 1$. We achieve that by making an ansatz on bounds on the correlation functions. We then prove (this part of the argument is relatively simple) that the ansatz is consistent with the equations that are obtained by applying the DLR equation to correlation functions of any order [similar to those in (1.14) for the two-body terms]. We then need to prove that the actual correlations satisfy the ansatz and this is done by a continuity argument starting from $\beta = 1 - \gamma^2$. In this part we use extensively Newmann's Gaussian inequalities and convexity properties of the pressure to prove uniqueness of the even correlation functions. With this information we can then modify the usual mean-field argument for the magnetization proving that also at $\beta_b(\gamma)$, $b < c^*$, it is equal to 0, thus obtaining Theorem 1. To prove Theorem 2 we use Aizenman's inequality on the four-point truncated correlation function to have a closed inequality for the two-point correlations, which can be solved till $b = c^*$ if ε^{-1} in (1.8) is large enough. By using an argument by contradiction we finally prove the existence of a nontrivial limit.

CONCLUDING REMARKS

Very schematically we present motivations and some of the main open problems we would like to study.

1. Unsatisfactory: we would like to work at γ small but nonzero and $\beta = \beta_{c^*} - a\gamma^2$, $a > 0$, and $a \to 0$. Aim: crossover from mean field to n.n. Ising critical exponent; see the discussion following (1.6).
2. Original motivation: Jona-Lasinio's proposal[10] of stochastic quantization via particle dynamics, in our case the Glauber dynamics, namely to derive stochastic PDEs whose invariant measure is the ϕ^4 Euclidean field theory. This is the dynamical analog of the approach to constructive ϕ^4 theories via Ising models, with the belief that, as in equilibrium, this may lead to substantial simplifications. Results in $d = 1$ have been already obtained.[4,8]
3. $d = 3$?

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